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A STUDY OF THE REPRESENTATIONS OF SL(2,$\mathbb{C}$) USING NONINFINITESIMAL METHODS

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A study of the representations of SL(2,$\mathbb{C}$) using noninfinitesimal methods

by

M.T. Kosters *)

ABSTRACT

The (nonunitary) pricipal series of SL(2,$\mathbb{C}$) is studied using explicit expresssions for its matrix elements with respect to certain K-bases.

In particular we find all subquotient representations of the principal series and all equivalences occurring among these. Using the subquotient theorem this allows us to classify the topologically completely irreducible representations of SL(2,$\mathbb{C}$) on a Banach space.

Calculating certain intertwining operators allows us to decide which representations can be made unitary by choosing a suitable inner product on the representation space.

KEY WORDS & PHRASES: *Representation theory of* SL(2,$\mathbb{C}$)*, principal series, canonical matrix elements, hypergeometric functions, equivalence of representations, unitarizability of representations.*

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1. INTRODUCTION

In this paper we study the representations of the group $G = SL(2,\mathbb{C})$ of 2×2 complex unimodular matrices. Our approach is almost identical to the approach in [6], a paper which deals with the representation theory of $SL(2,\mathbb{R})$.

In Section 2, we introduce the (nonunitary) principal series and derive a canonical basis consisting of K-finite vectors in the principal series representation spaces.

In Section 3 we calculate the matrix elements of the principal series representations with respect to these bases, and we use this to decide when the representations are irreducible, and to find the irreducible subquotients in the reducible case.

In Section 4 we determine the equivalences which exist between the subquotient representations we have found, and applying Harish-Chandra's subquotient theorem we use this to give a classification of the topologically completely irreducible representations of G. (Here our approach differs from that of [6], where completeness of the set of representations is proved by considering the eigenfunctions of the Casimir-operator.)

In Section 5 we determine which representations are unitarizable. We do this by determining when the representations of 4 are equivalent to their conjugate contragredient, and when the intertwining operator can be normalzed so that it becomes positive definite. This allows us to describe the unitary dual of G.

In Section 6 we give a (certainly incomplete) history of the problem in order to indicate what was already known.

All results in Sections 2,3,4 and 5 are obtained by noninfinitesimal methods, i.e. without using Lie-algebras (except of course that Lie- algebras are implicitly used when we apply the subquotient theorem). All reducibility properties and equivalences are found by using explicit expressions for the matrix elements of the principal series.

2. DEFINITION OF THE PRINCIPAL SERIES

$G = SL(2,\mathbb{C})$ is a six-dimensional semisimple noncompact Lie group with Iwasawa decomposition $G = KAN$, where

$$(2.1)\quad \begin{cases} K = SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = k(\alpha,\beta) \;\middle|\; |\alpha|^2+|\beta|^2 = 1 \right\}, \\[2ex] A = \left\{ \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} = a_t \;\middle|\; t \in \mathbb{R} \right\}, \\[2ex] N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = n_z \;\middle|\; z \in \mathbb{C} \right\}, \end{cases}$$

which means among other things that the mapping $K \times A \times N \to G$ given by $(k,a,n) \to (kan)$ is a real analytic diffeomorphism. Furthermore we have the decomposition $G = KAK$: every $g \in G$ can be written as $g = k_1 a k_2$ with $k_1, k_2 \in K$, $a \in A$, but this expression is not unique.

$$k(\alpha,\beta)a_t n_z = \begin{pmatrix} e^{t}\alpha & \cdots \\ -e^{t}\bar{\beta} & \cdots \end{pmatrix},$$

so if

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = k(\alpha,\beta)a_t n_z$$

then

$$(2.2)\quad \begin{cases} \alpha = \dfrac{g_{11}}{(|g_{11}|^2+|g_{21}|^2)^{\frac{1}{2}}}, \\[2ex] -\bar{\beta} = \dfrac{g_{21}}{(|g_{11}|^2+|g_{21}|^2)^{\frac{1}{2}}}, \\[2ex] e^{t} = (|g_{11}|^2+|g_{21}|^2)^{\frac{1}{2}}. \end{cases}$$

In particular,

$$a_t^{-1}(k(\alpha,\beta) = \begin{pmatrix} e^{-t}\alpha & \cdots \\ -e^{t}\bar{\beta} & \cdots \end{pmatrix} = k(\alpha^1,\beta^1)\; a_s n_z$$

implies:

$$(2.3)\quad \begin{cases} e^s = (e^{-2t}|\alpha|^2 + e^{2t}|\beta|^2)^{\frac{1}{2}}\;, \\[2ex] \alpha^1 = \dfrac{e^{-t}\alpha}{(e^{-2t}|\alpha|^2+e^{2t}|\beta|^2)^{\frac{1}{2}}}\;, \\[2ex] \beta^1 = \dfrac{e^{t}\beta}{(e^{-2t}|\alpha|^2+e^{2t}|\beta|^2)^{\frac{1}{2}}}\;. \end{cases}$$

If we parametrize K by coordinates θ,ϕ,ψ as follows:

$$(2.4)\qquad \begin{matrix} \alpha = \cos\theta\; e^{i\phi} \\ \beta = \sin\theta\; e^{i\psi} \end{matrix} \qquad (0 \le \theta \le {}^{\pi}/2\;,\; 0 \le \phi\;,\psi < 2\pi)$$

then the normalised Haar measure on K becomes:

$$(2.5)\qquad dk = \frac{1}{4\pi^2}\; \sin 2\theta\; \alpha\theta\; d\phi\; d\psi.$$

Let $M = \left\{ \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = u_\theta \,\middle|\, \theta \in \mathbb{R} \right\}$. M is the centralizer of A in K.

$\hat{M}$, the collection of finite-dimensional irreducible representations of M consists of the pairwise inequivalent elements ξ_k ($k \in \frac{1}{2}\mathbb{Z}$) given by

$$(2.6)\qquad \xi_k(u_\theta) = e^{-2ik\theta}\;.$$

(We write the elements of $\hat{M}$ in this way because it eliminates many factors by $\frac{1}{2}$ in our later calculations).

For $j = 0,\frac{1}{2},1,\frac{3}{2},..$ let V^j be the $(2j+1)$-dimensional complex vector space of polynomials in indeterminates z_1,z_2 with complex coefficients, homogeneous of degree $2j$.

K acts on V^j by:

(2.7) $$(\pi_j(k)f)(z_1,z_2) = f(k^{-1}(z_1,z_2)) \qquad (k \in K,\ f \in V^j)\ .$$

(We consider (z_1,z_2) as a *column* vector which can be multiplied from the left by $k^{-1} \in SU(2)$)

The representations π_j are irreducible, π_{j_1} and π_{j_2} are inequivalent if $j_1 \neq j_2$ and every finite dimensional irreducible representation of K is equivalent to a π_j. V^j has a basis $v_{-j}{}^j,\ v_{-j+1}{}^j, \ldots, v_{j-1}{}^j,\ v_j{}^j$, given by

(2.8) $$v_\ell{}^j(z_1,z_2) = z_1{}^{j-\ell} z_2{}^{j+\ell}\ .$$

If we define an inner product (,) on V^j by

$$(v_{\ell_1}{}^j, v_{\ell_2}{}^j) = \delta_{\ell_1,\ell_2}\,(j-\ell_1)!\,(j+\ell_1)!$$

then π_j is unitary.

$$\pi_j(u_\theta)v_\ell{}^j = e^{-2i\ell\theta} v_\ell{}^j ,$$

so, as a representation of M, π_j splits as a direct sum of ξ_ℓ's ($\ell = -j, -j+1, \ldots, j-1, j$), each of these occurring exactly once, so

(2.9) $$\dim \mathrm{Hom}_M\,(V^j, V^{\xi_k}) = \begin{cases} 1 \text{ if } k-j \in \mathbb{Z} \quad |k| \leq j\ , \\ 0 \text{ otherwise} \quad , \end{cases}$$

where $V^{\xi_k} = \mathbb{C}$ is the onedimensional vector space on which the representation ξ_k is realised, $\mathrm{Hom}_M\,(V^j, V^{\xi_k})$ is the space of M-intertwining linear mappings $A: V^j \to V^{\xi_k}$. If $\dim \mathrm{Hom}_M(V^j, V^{\xi_k}) = 1$, then $\mathrm{Hom}_M(V^j, V^{\xi_k})$ is spanned by $A_{k,j}: V^j \to V^{\xi_k} = \mathbb{C}$ given by

(2.10) $$A_{k,j}\, v_\ell{}^j = \delta_{k,\ell}\ .$$

Now we are ready to define the principal series of representations

$\pi_{\xi_k,z}$ ($\xi_k \in \hat{M}$, $z \in \mathbb{C}$) of G on Hilbert spaces $H_{\xi_k,z}$: For $\xi_k \in \hat{M}$, $z \in \mathbb{C}$ let $C_{\xi_k,z}(G)$ be the space of continuous functions $f : G \to V^{\xi_k}$ satisfying the relation

$$(2.11) \qquad f(g\ man) = \xi_k(m^{-1})\ e^{-2(z+1)\log a}\ f(g) \quad \forall g \in G,\ m \in M,\ a \in A,\ n \in N,$$

where $\log: A \to \mathbb{R}$ is defined by $\log a_t = t$.
G acts on $C_{\xi_k,z}(G)$ by

$$(2.12) \qquad (\pi_{\xi_k,z}(g)f)(x) = f(g^{-1}x) \quad (x,g \in G)$$

Define an inner product (,) on $C_{\xi_k,z}(G)$ by

$$(2.13) \qquad (f,g) = \int_K f(k)\overline{g(k)}\,dk$$

and let $H_{\xi_k,z}$ be the closure of $C_{\xi_k,z}$ with respect to this inner product. The operators $\pi_{\xi_k,z}(g)$ can be extended uniquely to bounded operators on the separable Hilbert space $H_{\xi_k,z}$ which we shall also denote by $\pi_{\xi_k,z}(g)$. Then $\pi_{\xi_k,z}$ is a strongly continuous representation of G on $H_{\xi_k,z}$, $\pi_{\xi_k,z}$ is unitary, if and only if z is purely imaginary. $\pi_{\xi_k,z}$ is the representation of G induced by the onedimensional representation $\mu_{\xi_k,z}$ of the standard minimal parabolic subgroup B = MAN defined by

$$\mu_{\xi_k,z}(man) = \xi_k(m)\ e^{2z\ \log a}.$$

The restriction of $\pi_{\xi_k,z}$ to K splits as a unitary direct sum of irreducible representations $H^j_{\xi_k,z}$ equivalent to V^j, each j occurring at most once. To be more precise:

$$(2.14) \qquad H_{\xi_k,z} = \overline{\bigoplus_{\substack{j \ge |k| \\ j-k \in \mathbb{Z}}} H^j_{\xi_k,z}} \quad \text{as a K-representation}.$$

This follows immediately from (2.9) and the Frobenius reciprocity theorem if we observe that the restriction of $\pi_{\xi_k,z}$ to K is just the representation of K induced by the representation ξ_k of its subgroup M (this is because of the Iwasawa decomposition). We now exhibit non-trivial elements of $H^j_{\xi_k,z}$: For $A \in \mathrm{Hom}_M(V^j, V^{\xi_k})$, $v \in V^j$ let $f(kan) = e^{-2(z+1)\log a} A(\pi_j(k^{-1})v)$, then $f \in H^j_{\xi_k,z}$. In fact the mapping $v \otimes A \to f$ defines a K-intertwining isomorphism $V^j \otimes \mathrm{Hom}_M(V^j, V^{\xi_k}) \to H^j$ if we let K act trivially on $\mathrm{Hom}_M(V^j, V^{\xi_k})$. Now take $v = v_\ell^j.((j-\ell)!(j+\ell)!)^{-\frac{1}{2}}$, $A = A_{k,j}.((j-k)!(j+k)!)^{\frac{1}{2}}$ (see (2.8) and (2.10)).

$$(\pi_j(k(\alpha\ \beta)^{-1}) v_\ell^j)(z_1,z_2) = (\alpha z_1+\beta z_2)^{j-\ell}(-\bar\beta z_1+\bar\alpha z_2)^{j+\ell} =$$

$$\sum_{\kappa=0}^{j-\ell}\sum_{\nu=0}^{j+\ell}\binom{j-\ell}{\kappa}\binom{j+\ell}{\nu}\alpha^\kappa\beta^{j-\ell-\kappa}z_1^{\ \kappa}z_2^{\ j-\ell-\kappa}(-\bar\beta)^\nu\bar\alpha^{j+\ell-\nu}z_1^{\ \nu}z_2^{\ j+\ell-\nu} =$$

$$\ldots + \sum_{\kappa=\max(0,-\ell-k)}^{\min(j-\ell,j-k)}\binom{j-\ell}{\kappa}\binom{j+\ell}{j-k-\kappa}\alpha^\kappa\beta^{j-\ell-\kappa}(-\bar\beta)^{j-k-\kappa}\bar\alpha^{\ell+k+\kappa}z_1^{j-k}z_2^{j+k}+\ldots\ .$$

This gives us an element $f^j_{\ell,k} \in H^j_{\xi_k,z}$:

$$f^j_{\ell,k}(k(\alpha,\beta)) = \left(\frac{(j-k)!(j+k)!}{(j-\ell)!(j+\ell)!}\right)^{\frac{1}{2}} . \tag{2.15}$$

$$\cdot \sum_{\kappa=\max(0,-\ell-k)}^{\min(j-\ell,j-k)}\binom{j-\ell}{\kappa}\binom{j+\ell}{j-k-\kappa}\alpha^\kappa\beta^{j-\ell-\kappa}(-\bar\beta)^{j-k-\kappa}\bar\alpha^{\ell+k+\kappa}\ .$$

Observe that the functions $f^j_\ell(k)$ are matrix elements of the representation π_j defined by (2.7) with respect to an orthonormal basis:

$$f^j_{\ell,k}(k) = (\pi_j(k^{-1})((j-\ell)!(j+\ell)!)^{-\frac{1}{2}}v_\ell^{\ j}, ((j-k)!(j+k)!)^{-\frac{1}{2}}v_k^{\ j})$$

$$= (((j-\ell)!(j+\ell)!)^{-\frac{1}{2}}v_\ell^{\ j},\ \pi_j(k)((j-k)!(j+k)!)^{-\frac{1}{2}}\ v_k^{\ j})$$

3. CALCULATION OF THE MATRIX ELEMENTS OF $\pi_{\xi_k,z}$. REDUCIBILITY PROPERTIES.

We calculate the matrix elements of $\pi_{\xi_k,z}$ with respect to the orthogonal basis consisting of the $f^{j}_{\ell,k}$ and then use Theorem 3.2 of [6] to deduce which $\pi_{\xi_k,z}$ are irreducible and to determine the irreducible subquotient representations in case $\pi_{\xi_k,z}$ is reducible. Let

$$\pi_{\xi_k,z;j_1\ell_1;j_2,\ell_2}(g) = (\pi_{\xi_k,z}(g)f^{j_2}_{\ell_2,k},\ f^{j_1}_{\ell_1,k}) .$$

By (2.3), (2.11), (2.12), (2.13) and (2.15) we get:

$$\begin{aligned}(3.1)\qquad &\pi_{\xi_k,z;j_1,\ell_1;j_2,\ell_2}(a_t) = \\ &= \int_K (e^{-2t}|\alpha|^2 + e^{2t}|\beta|^2)^{-(z+1)} f^{j_2}_{\ell_2,k}k(\alpha^1,\beta^1)\overline{f^{j_1}_{\ell_1,k}(k(\alpha,\beta))}dk(\alpha,\beta)\end{aligned}$$

Some simple considerations show us that a large number of these functions must be zero : M acts on $f^{j}_{\ell,k}$ as ξ_ℓ. The elements of A commute with those of M, so M acts on $\pi_{\xi_k,z}(a_t)f^{j}_{\ell,k}$ again as ξ_ℓ, so

$$(3.2)\qquad (\pi_{\xi_k,z}(a_t)f^{j_2}_{\ell_2,k},\ f^{j_1}_{\ell_1,k}) = 0 \quad \text{if } \ell_1 \neq \ell_2 ,$$

so from now on we assume $\ell_1 = \ell_2 = \ell$.

$$(3.3)\qquad e^{-2t}|\alpha|^2 + e^{2t}|\beta|^2 = e^{-2t}(1-|\beta|^2) + e^{2t}|\beta|^2 = e^{-2t}(1-(1-e^{4t})|\beta|^2).$$

We expand this in a power series in $(1-e^{4t})$ and change the order of summation and integration to find an expression for the matrix elements involving power series in $(1-e^{4t})$. (These, and all the following expressions only hold for sufficiently small values of $1-e^{4t}$, but as the matrix elements are real analytic functions of t it suffices to know them in a neighbourhood of t=0. Thus, from now on we shall tacitly assume that in all expressions t is sufficiently small.)

Write

$$F^{j_2}_{\ell,k,\kappa,\nu}(k(\alpha,\beta)) = (-1)^{j_2-k-\kappa}\left(\frac{(j_2-k)!(j_2+k)!}{(j_2-\ell)!(j_2+\ell)!}\right)^{\frac{1}{2}} \cdot \tag{3.4}$$

$$\cdot \binom{j_2-\ell}{\kappa}\binom{j_2+\ell}{j_2-k-\kappa}\alpha^{\kappa}\beta^{j_2-\ell-\kappa+\nu}\bar{\beta}^{j_2-k-\kappa+\nu}\cdot\bar{\alpha}^{-\ell+k+\kappa}$$

($F^{j_2}_{\ell,k,\kappa,\nu}$ is the κ-th term of $f^{j_2}_{\ell,k}$ multiplied by $(|\beta|^2)^{\nu}$) and

$$C^{j_1,j_2;\ell;k}_{\nu,\kappa} = (F^{j_2}_{\ell,k,\kappa,\nu}, f^{j_1}_{\ell,k}) . \tag{3.5}$$

Then

$$\pi_{\xi_k,z;j_1,\ell;j_2,\ell}(a_t) = \tag{3.6}$$

$$= e^{2t(z+1+2j_2-\ell-k)} \sum_{\kappa=\max(0,-\ell-k)}^{\min(j_2-\ell,j_2-k)} e^{-4\kappa t} .$$

$$\cdot \sum_{\nu=0}^{\infty} C^{j_1,j_2;\ell;k}_{\nu,\kappa} \frac{(z+1+j_2)_{\nu}}{\nu!} (1-e^{4t})^{\nu} .$$

Now we claim that

$$C^{j_1,j_2;\ell;k}_{\nu,\kappa} = 0 \text{ for } \nu \le j_1 - j_2 - 1 . \tag{3.7}$$

Indeed, $F^{j_2}_{\ell,k,\kappa,\nu}$ is contained in the space W of polynomials in $\alpha,\bar{\alpha},\beta,\bar{\beta}$, of degree $\le 2j_2 + 2\nu$. W is invariant under the left regular action of K. As a representation of M,W splits into one-dimensional subrepresentations ξ_k ($k = -j_2-\nu, -j_2-\nu+1,\ldots,j_2+\nu$). $f^{j_1}_{\ell}$ is contained in a left K-invariant irreducible subspace which contains ξ_{j_1} as an M-representation. Hence W cannot contain the representation π_{j_1} of K. Now (3.7) follows from the orthogonality relations for the left-regular representation of K.

Next we show that

$$(3.8)\qquad C^{j_1,j_2;j_2;k}_{j_1-j_2,0} \neq 0 \quad \text{for } j_2 \leq j_1 .$$

We do this by giving quite explicit expressions for certain matrix elements, which we shall also need for the study of unitarizability of the representations.

$$(3.9)\qquad \pi_{\xi_k,z;j_1,j_2;j_2,j_2}(a_t) =$$

$$= e^{2t(z+1+j_2-k)} \sum_{\nu=0}^{\infty} C^{j_1,j_2;j_2,k}_{\nu,0} \frac{(z+1+j_2)_\nu}{\nu!} (1-e^{4t})^\nu$$

with

$$C^{j_1,j_2;j_2;k}_{\nu,0} =$$

$$= (-1)^{j_1+j_2-2k} \binom{2j_2}{j_2-k} \left(\frac{(j_2-k)!(j_2+k)!(j_1-k)!(j_1+k)!}{(2j_2)!(j_1-j_2)!(j_1+j_2)!} \right)^{\frac{1}{2}} .$$

$$\cdot \sum_{\kappa=0}^{j_1-j_2} \binom{j_1-j_2}{\kappa} \binom{j_1+j_2}{j_1-k-\kappa} (-1)^\kappa \int_K (\alpha\bar{\alpha})^{j_2+k+\kappa} (\beta\bar{\beta})^{j_1-k-\kappa+\nu} dk(\alpha,\beta) ,$$

because of (2.15),(3.4) and (3.5).

For the integral over K we can write

$$\int_{\theta=0}^{\pi/2} (\cos^2\theta)^{j_2+k+\kappa} (\sin^2\theta)^{j_1-k-\kappa+\nu}\, d\sin^2\theta =$$

$$\int_{t=0}^{1} (1-t)^{j_2+k+\kappa}\, t^{j_1-k-\kappa+\nu}\, dt =$$

$$\frac{\Gamma(j_2+k+\kappa+1)\,\Gamma(j_1-k-\kappa+\nu+1)}{\Gamma(j_1+j_2+\nu+2)} ,$$

where we used (2.5) and the well-known integral representation for the beta-function:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{(z+w)} = \int_0^1 t^{z-1}(1-t)^{w-1}dt \qquad ([2],\S1.5(1)).$$

Substitution of this last identity into the expression for $C^{j_1,j_2;j_2,k}_{\nu,0}$ and introduction of the new summation variable $\kappa^1 = j_1-j_2-\kappa$ yields:

(3.10) $$C^{j_1,j_2;j_2,k}_{\nu,0} = \binom{2j_2}{j_2-k} \frac{(j_1+j_2)!}{(j_2-k)!} \frac{(j_2-k+\nu)!}{(j_1+j_2+\nu+1)!} \cdot$$

$$\cdot \left(\frac{(j_2-k)!(j_2+k)!(j_1-k)!(j_1+k)!}{(2j_2)!(j_1-j_2)!(j_1+j_2)!} \right)^{\frac{1}{2}} {}_2F_1(j_2-j_1, j_2-k+\nu+1;\ j_2-k+1;1) =$$

$$= \binom{2j_2}{j_2-k} \frac{(j_1+j_2)!}{(j_2-k)!} \frac{(j_2-k+\nu)!}{(j_1+j_2+\nu+1)!} \left(\frac{(j_2-k)!(j_2+k)!(j_1-k)!(j_1+k)!}{(2j_2)!(j_1-j_2)!(j_1+j_2)!} \right)^{\frac{1}{2}} \cdot$$

$$\cdot \frac{(-\nu)_{j_1-j_2}}{(j_2-k+1)_{j_1-j_2}} ,$$

where we used the relation

$$_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n} \quad (n=0,1,2,\dots,c \neq 0,-1,-2,\dots,-n+1).$$

This proves (3.8).

For later use in §5 we write down an expression for the matrix elements $\pi_{\xi_k,z,j_1,j_2;j_2,j_2}(a_t)$ which follows from the above calculations ($j_2 \leq j_1$):

(3.11) $$\pi_{\xi_k,z;j_1,j_2;j_2,j_2}(a_t) =$$

$$= (-1)^{j_2-j_1} \left(\frac{(2j_2)!(j_1+j_2)!(j_1-k)!(j_1+k)!}{(j_1-j_2)!(j_2-k)!(j_2+k)!}\right)^{\frac{1}{2}} \frac{(z+1+j_2)_{j_1-j_2}}{(2j_1+1)!} \,.$$

$$. \, e^{2t(z+j_2-k+1)} (1-e^{4t})^{j_1-j_2} {}_2F_1(z+1+j_1,j_1-k+1;2j_1+2;1-e^{4t}).$$

Using (3.7), (3.8) and theorem 3.2 of [6] we find the reducibility properties of the $\pi_{\xi_k,z}$. Denote the (j_1,j_2) generalized matrix element (see [6]p.7) by π_{ξ_k,z,j_1,j_2}, then it follows from (3.6),(3.7) and (3.8) that for $j_1 \geq j_2$:

(3.12) $$\pi_{\xi_k,z,j_1,j_2}(a_t) \equiv 0 \Longleftrightarrow (z+1+j_2)_{j_1-j_2} = 0 \,.$$

For the case $j_1 \leq j_2$ we use the fact that the conjugate contragredient of $\pi_{\xi_k,z}$ is equal to $\pi_{\xi_k,-\bar{z}}$, so

$$\pi_{\xi_k,z,j_1,\ell_1,j_2,\ell_2}(a_t) = \overline{\pi_{\xi_k,-\bar{z};j_2,\ell_2,j_1,\ell_1}(a_{-t})} \quad .$$

This shows that for $j_2 \geq j_1$

$$(3.13) \qquad \pi_{\xi_k,z,j_1,j_2}(a_t) \equiv 0 \iff (z-j_2)_{j_2-j_1} = 0 \ .$$

Because of the decomposition G = KAK: $\pi_{\xi_{k,z,j_1,j_2}}(a_t) \equiv 0 \Leftrightarrow \pi_{\xi_k,z,j_1,j_2}(g) \equiv 0$ on G.

Applying Theorem 3.2 of [6] we obtain a theorem analogous to Theorem 3.4 of [6] (an asterisk at some place in the diagrams occurring in the theorem means that all generalized matrix elements π_{ξ_k,z,j_1,j_2} corresponding to that block are nonzero):

THEOREM 1. *Depending on* ξ_k *and* z *the representation* $\pi_{\xi_k,z}$ *of* SL(2,ℂ) *has the following irreducible subquotient representations and subrepresentations:*

a. $\pi_{\xi_k,z}$ *is irreducible if and only if* $z \neq \pm(|k|+j)$ *for every positive integer* j.

b. $z = -(|k|+j)$ *for some positive integer* j:

π_{ξ_k,z,j_1,j_2}:

$j_1 \downarrow \quad \vec{j_2}$	$[|k|, -z-1]$	$[-z,\infty)$
$[|k|, -z-1]$	*	*
$[-z,\infty)$	0	*

Irreducible finite-dimensional subrepresentation on

$$\bigoplus_{j=|k|}^{-z-1} H^{j}_{\xi_k,z} \quad ;$$

irreducible infinite-dimensional subquotient representation but not subrepresentation on

$$\overline{\bigoplus_{j=-z}^{\infty} H^{j}_{\xi_k,z}} \; .$$

c. $z = |k|+j$ *for some positive integer* j

$$\pi_{\xi_k,z,j_1,j_2} :$$

$j_1 \downarrow \quad j_2 \rightarrow$	$[\,|k|,\ z-1]$	$[z,\infty)$
$[\,|k|,z-1]$	*	0
$[z,\infty)$	*	*

Irreducible finite-dimensional subquotient representation but not subrepresentation on

$$\bigoplus_{j=|k|}^{z-1} H^{j}_{\xi_k,z} \quad ;$$

irreducible infinite-dimensional subrepresentation on

$$\overline{\bigoplus_{j=z}^{\infty} H^{j}_{\xi_k,z}} \; .$$

(Here $[a,b]$ means $\{a,a+1,a+2,\ldots,b-2,b-1,b\}$, so the summation index j increases in steps of 1.)

4. EQUIVALENCES OCCURRING BETWEEN THE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

We want to determine which of the irreducible representations of $SL(2,\mathbb{C})$ we have found are Naimark-equivalent to each other. We want to apply Theorem 4.6 of [6] and therefore need to compute diagonal matrix elements. In what follows we shall denote by $\tau_{\xi_k,z}$ and $\sigma_{\xi_k,z}$ the infinite-respectively finite-dimensional proper subquotient representation of $\pi_{\xi_k,z}$ in case $\pi_{\xi_k,z}$ is reducible. τ_{ξ_k,z,j_1,j_2} etc. will have the obvious meaning.

We shall first show that $\pi_{\xi_{-k},-z} \simeq \pi_{\xi_k,z}$, in case $\pi_{\xi_k,z}$ is irreducible, and $\sigma_{\xi_{-k},-z} \simeq \sigma_{\xi_k,z}$, $\tau_{\xi_{-k},-z} \simeq \tau_{\xi_k,z}$ in case $\pi_{\xi_k,z}$ is reducible. This will be proved when we have shown that

$$(4.1) \qquad \pi_{\xi_k,z;j,\ell;j,\ell}(a_t) = \pi_{\xi_{-k},-z;j;\ell;j,\ell}(a_t) \quad \forall\ k,z,j,\ell$$

(by [6], theorem 4.6, and (3.2)). Now, from (2.15),(3.4),(3.5) and (3.6)

$$(4.2) \qquad \pi_{\xi_k,z;j,\ell;j,\ell}(a_t) = e^{2t(z+1+2j-\ell-k)} \frac{(j-k)!\,(j+k)!}{(j-\ell)!\,(j+\ell)!} \cdot$$

$$\cdot \sum_{\kappa_1} e^{-4\kappa_1 t} \binom{j-\ell}{\kappa_1}\binom{j+\ell}{j-k-\kappa_1} \cdot$$

$$\cdot \sum_{\kappa_2} \binom{j-\ell}{\kappa_2}\binom{j+\ell}{j-k-\kappa_2} \sum_{\nu=0}^{\infty} C^{\ell,k,j}_{\nu,\kappa_1,\kappa_2} \cdot \frac{(z+1+j)_\nu}{\nu!} (1-e^{4t})^{\nu}$$

with

$$(4.3) \qquad C^{\ell,k,j}_{\nu,\kappa_1,\kappa_2} = (-1)^{\kappa_1+\kappa_2} \int_K (\alpha\bar{\alpha})^{\ell+k+\kappa_1+\kappa_2} (\beta\bar{\beta})^{2j-\ell-k+\nu-\kappa_1-\kappa_2} dk(\alpha,\beta) =$$

$$(-1)^{\kappa_1+\kappa_2} \int_{\theta=0}^{\pi/2} (1-\sin^2\theta)^{\ell+k+\kappa_1+\kappa_2} (\sin^2\theta)^{2j-\ell-k+\nu-\kappa_1-\kappa_2} d\,\sin^2\theta =$$

$$= (-1)^{\kappa_1+\kappa_2} \frac{\Gamma(\ell+k+\kappa_1+\kappa_2+1)\Gamma(2j-\ell-k+\nu-\kappa_1-\kappa_2+1)}{\Gamma(2j+\nu+2)}$$

Substituting (4.3) in (4.2) enables us to express the power series as hypergeometric functions:

(4.4) $$\pi_{\xi_k \cdot z;j,\ell;j,\ell}(a_t) = e^{2t(z+1+2j-\ell-k)} \frac{(j-k)!(j+k)!}{(j-\ell)!(j+\ell)!} \cdot$$

$$\cdot \sum_{\kappa_1} e^{-4\kappa_1 t} \binom{j-\ell}{\kappa_1}\binom{j+\ell}{j-k-\kappa_1} \sum_{\kappa_2} \binom{j-\ell}{\kappa_2}\binom{j+\ell}{j-k-\kappa_2} \cdot$$

$$\cdot \frac{(-1)^{\kappa_1+\kappa_2}}{(2j+1)\binom{2j}{\ell+k+\kappa_1+\kappa_2}} \; {}_2F_1(z+j+1, 2j-\ell-k-\kappa_1-\kappa_2+1; 2j+2; 1-e^{4t}).$$

Using formula (2), §2.9 of [2] for the transformation of the hypergeometric function we find that expression (4.4) is equal to

(4.5) $$e^{2t(-z+1+2j-\ell+k)} \frac{(j-k)!(j+k)!}{(j-\ell)!(j+\ell)!} \sum_{\kappa_1} \binom{j-\ell}{j-\ell-\kappa_1}\binom{j+\ell}{j+k-(j-\ell-\kappa_1)}$$

$$\sum_{\kappa_2} \binom{j-\ell}{j-\ell-\kappa_2}\binom{j+\ell}{j+k-(j-\ell-\kappa_2)} \frac{(-1)^{(j-\ell-\kappa_1)+(j-\ell-\kappa_2)}}{(2j+1)\binom{2j}{\ell-k+(j-\ell-\kappa_1)+(j-\ell-\kappa_2)}} \cdot$$

$$\cdot \; {}_2F_1(-z+j+1, 2j-\ell+k-(j-\ell-\kappa_1)-(j-\ell-\kappa_2)+1; 2j+2; 1-e^{4t}).$$

If we write κ_2 instead of $j-\ell-\kappa_1$ and κ_1 instead of $j-\ell-\kappa_2$ in the last expression we find that it is equal to $\pi_{\xi_{-k},-z;j,\ell;j,\ell}(a_t)$ as we had claimed before.

Now we ask whether the equivalences we have just established are the only ones occurring between the various subquotientrepresentations of the principal series. By (4.4):

$$\pi_{\xi_k,z;j,j;j,j}(a_t) = C(1+c_{k,z,j}(1-e^{4t})+d_{k,z,j}(1-e^{4t})^2+..)$$

for a certain normalization constant $C \neq 0$, with

$$(4.6) \qquad c_{k,z,j} = \frac{-zk}{2j+2} .$$

So if two subquotient representations of $\pi_{\xi_k,z}$ respectively $\pi_{\xi_k^1,z^1}$ equivalent, then $zk = z^1k^1$. Comparing K-contents we conclude:

$$\pi_{\xi_k,z},\ \pi_{\xi_{k^1},z^1} \text{ irreducible, } \pi_{\xi_k,z} \simeq \pi_{\xi_{k^1},z^1} \Rightarrow (k,z) = (k^1,z^1) \text{ or } (k,z) = (-k^1,-z^1) ;$$

$$\tau_{\xi_k,z} \simeq \tau_{\xi_{k^1},z^1} \Rightarrow (k,z) = (k^1,z^1) \text{ or } (k,z) = (-k^1,-z^1) ;$$

$$\sigma_{\xi_k,z} \simeq \sigma_{\xi_{k^1},z^1} \Rightarrow (k,z) = (k^1,z^1) \text{ or } (k,z) = (-k^1,-z^1) .$$

Furthermore $\sigma_{\xi_k,z}$ can clearly never be equivalent to a $\tau_{\xi_{k^1},z^1}$ or a $\pi_{\xi_{k^1},z^1}$, so the only possible equivalences except the ones we have already found are between $\pi_{\xi_k,z}$ and $\tau_{\xi_z,k}$ (and then also $\tau_{\xi_{-z},-k}$) for $|z| < |k|$, $z-k \in \mathbb{Z}$. We claim that these equivalences do indeed exist. Because of the equivalences we have already proved we may assume $k > 0$ and then need only prove $\pi_{\xi_k,z} \simeq \tau_{\xi_z,k}$ for $|z| < k$, $z-k \in \mathbb{Z}$. So we have to prove that the matrix elements $\pi_{\xi_k,z;k,\ell;k,\ell}(a_t)$ and $\pi_{\xi_z,k;k,\ell;k,\ell}(a_t)$ are proportional.

To facilitate the calculations we derive some symmetry properties of the functions $f^j_{\ell,k}$, (see also [16], chapter III, §3.6), and use these to derive symmetry properties of the matrix elements of $\pi_{\xi_k,z}$:

LEMMA 1. *a.* $f^j_{\ell,k}(k(\alpha,\beta)) = f^j_{k,\ell}(k(\alpha,-\bar{\beta}))$.

b. $f^j_{\ell,k}(k(\bar{\alpha},-\bar{\beta})) = f^j_{-\ell,-k}(k(\alpha,\beta))$.

PROOF. Write the normalized basis vector $((j-\ell)!(j+\ell)!)^{-\frac{1}{2}}v_\ell{}^j$ of V^j as $w_\ell{}^j$.

a.

$$f^j_{\ell,k}(k(\alpha,\beta)) = (\pi_j(k(\alpha,\beta)^{-1})w_\ell{}^j, w_k{}^j) =$$

$$(w_\ell{}^j, \pi_j(k(\alpha,\beta))w_k{}^j) = \overline{(\pi_j(k(\bar{\alpha},-\beta)^{-1})w_k{}^j, w_\ell{}^j)} =$$

$$\overline{f^j_{k,\ell}(k(\bar{\alpha},-\beta))} = f^j_{k,\ell}(k(\alpha,-\bar{\beta}))$$

because the coefficients of the polynomial (2.15) are real.

b.

$$\pi_j(k(0,-i))w_\ell{}^j = i^{2j}w_{-\ell}{}^j.$$

$$f^j_{\ell,k}(k(\bar{\alpha},-\bar{\beta})) = (\pi_j(k(\bar{\alpha},-\bar{\beta})^{-1})w_\ell{}^j, w_k{}^j) =$$

$$(\pi_j(k(0,i))\pi_j(k(\alpha,\beta)^{-1})\pi_j(k(0,-i))w_\ell{}^j, w_k{}^j) =$$

$$(\pi_j(k(\alpha,\beta)^{-1})\ \pi_j(k(0,-i))w^j_\ell, \pi_j(k(0,-i))w_k{}^j) =$$

$$(\pi_j(k(\alpha,\beta))^{-1}w^j_{-\ell}, w_{-k}{}^j) = f^j_{-\ell,-k}(k(\alpha,\beta))$$

LEMMA 2.

$$\pi_{\xi_k,z;j_1,\ell;j_2,\ell}(a_t) = \pi_{\xi_\ell,z;j_1,k;j_2,k}(a_t) = \pi_{\xi_{-k},z;j_1,-\ell;j_2,-\ell}(a_t).$$

PROOF.

$$(\pi_{\xi_k,z}(a_t)f^j_{\ell,k})(k(\alpha,\beta)) = (e^{-2t}|\alpha|^2+e^{2t}|\beta|^2)^{-(z+j+1)} \cdot$$

$$\cdot\ f^j_{\ell,k}\ (k(e^{-t}\alpha,e^t\beta))\ .$$

By lemma 1 *a*:

$$\pi_{\xi_k,z;j_1,\ell;j_2,\ell}(a_t) = (\pi_{\xi_k,z}(a_t)f^{j_2}_{\ell,k}\ ,\ f^{j_1}_{\ell,k}) =$$

$$\int_K (e^{-2t}|\alpha|^2+e^{2t}|\beta|^2)^{-(z+j+1)} f^{j_2}_{\ell,k}(k(e^{-t}\alpha,e^t\beta))\ \overline{f^{j_1}_{\ell,k}k(\alpha,\beta)}dk(\alpha,\beta) =$$

$$\int_K (e^{-2t}|\alpha|^2+e^{2t}|\beta|^2)^{-(z+j+1)} f^{j_2}_{k,\ell}(k(e^{-t}\alpha,-e^{t}\bar{\beta}))\overline{f^{j_1}_{k,\ell}k(\alpha,-\bar{\beta})dk(\alpha,\beta)} =$$

$$(\pi_{\xi_\ell,z}(a_t)f^{j_2}_{k,\ell},f^{j_1}_{k,\ell}) = \pi_{\xi_\ell,z;j_1,k;j_2,k}(a_t).$$

The second equality is proved similarly by using Lemma 1 *b*.

Now we derive a slightly different expression for the matrix elements which has the convenience that the exponential factors are independent of the summation variables κ_1 and κ_2. If $\ell \geq -k$, then we see:

$$f^j_{\ell,k}(k(\alpha,\beta)) = \beta^{j-\ell}(-\bar{\beta})^{j-k}\ \bar{\alpha}^{\ell+k} \left(\frac{(j+\ell)!(j+k)!}{(j-\ell)!(j-k)!}\right)^{\frac{1}{2}} \cdot \frac{1}{(\ell+k)!}\ \cdot \tag{4.7}$$

$${}_2F_1(-j+\ell,-j+k;\ell+k+1;\ \frac{-|\alpha|^2}{|\beta|^2}\ ).$$

Now use formula (2), §2.9 of [2] and the fact that $|\alpha|^2 + |\beta|^2 = 1$ to transform this into a ${}_2F_1$ function of argument $|\alpha|^2$ and then expand this expression again in powers of α and $\bar{\alpha}$.

The result is:

$$(4.8)\qquad f^{j}_{\ell,k}(k(\alpha,\beta)) = \left(\frac{(j-k)!(j+k)!}{(j-\ell)!(j+\ell)!}\right)^{\frac{1}{2}} \cdot \sum_{\kappa=0}^{j-\ell} (-1)^{j-k-\kappa} \binom{j-\ell}{\kappa}\binom{\ell+j+\kappa}{\ell+k+\kappa} \alpha^{\kappa}\bar{\alpha}^{\ell+k+\kappa}\bar{\beta}^{\ell-k} .$$

Proceeding analogously for the case $\ell \le -k$ we find that in both cases:

$$(4.9)\qquad f^{j}_{\ell,k}(k(\alpha,\beta)) = \left(\frac{(j-k)!(j+k)!}{(j-\ell)!(j+\ell)!}\right)^{\frac{1}{2}} \cdot \sum_{\kappa=\max(0,-\ell-k)}^{j-\ell} (-1)^{j-k-\kappa} \binom{j-\ell}{\kappa}\binom{\ell+j+\kappa}{\ell+k+\kappa} \alpha^{\kappa}\bar{\alpha}^{\ell+k+\kappa}\bar{\beta}^{\ell-k} .$$

This gives us a new expression for the matrix elements:

$$(4.10)\qquad \pi_{\xi_k,z;j,\ell;j,\ell}(a_t) = e^{2t(z+1+\ell-k)} \frac{(j-k)!(j+k)!}{(j-\ell)!(j+\ell)!} \cdot \sum_{\kappa_1,\kappa_2=\max(0,-\ell-k)}^{j-\ell} (-1)^{\kappa_1+\kappa_2} \binom{j-\ell}{\kappa_1}\binom{\ell+j+\kappa_1}{\ell+k+\kappa_1} \cdot \binom{j-\ell}{\kappa_2}\binom{\ell+j+\kappa_2}{\ell+k+\kappa_2} \frac{(\ell+k+\kappa_1+\kappa_2)!(\ell-k)!}{(2\ell+\kappa_1+\kappa_2+1)!} \cdot {}_2F_1(z+1+\ell+\kappa_1, \ell-k+1; 2\ell+\kappa_1+\kappa_2+2; 1-e^{4t})$$

for $\ell \ge k$, and a similar expression for $\ell \le k$. For $\ell \ge \pm z$ this yields

(4.11) $$\pi_{\xi_z,k;k,\ell;k,\ell}(a_t) = e^{2t(-z+1+k+\ell)} \frac{(k-z)!(k+z)!}{(k-\ell)!(k+\ell)!} \cdot$$

$$\cdot \sum_{\nu=0}^{\infty} \frac{C_\nu^{z,k,\ell}}{\nu!} (1-e^{4t})^\nu ,$$

with

$$C_\nu^{z,k,\ell} = \sum_{\kappa_1,\kappa_2=0}^{k-\ell} (-1)^{\kappa_1+\kappa_2} \binom{k-\ell}{\kappa_1}\binom{\ell+k+\kappa_1}{\ell+z+\kappa_1}\binom{k-\ell}{\kappa_2}\binom{\ell+k+\kappa_2}{\ell+z+\kappa_2} \cdot$$

$$\cdot \frac{(\ell+z+\kappa_1+\kappa_2)!}{(2\ell+\kappa_1+\kappa_2+1)!}(\ell-z)! \frac{(k+\ell+1+\kappa_1)_\nu(\ell-z+1)_\nu}{(2\ell+\kappa_1+\kappa_2+2)_\nu} .$$

Now sum over κ_1 to find:

(4.12) $$C_\nu^{z,k,\ell} = \frac{(k-\ell)!}{((k-z)!)^2} \frac{(\ell-z+\nu)!(k+\ell+\nu)!}{(\ell+z)!} \sum_{\kappa_2} \frac{(-1)^{\kappa_2}}{\kappa_2!} \frac{(k+\ell+\kappa_2)!}{(k-\ell-\kappa_2)!(2\ell+\kappa_2+\nu+1)!}$$

$$\cdot {}_3F_2\left(\begin{matrix}k+\ell+\nu+1, & \ell+z+\kappa_2+1, & \ell-k; & 1\\ \ell+z+1 & , & 2\ell+\kappa_2+\nu+2 & \end{matrix}\right) .$$

Now the ${}_3F_2$ of unity argument in this expression is terminating because $\ell-k \le 0$, and Saalschützian, so it can be summed in terms of Γ-functions. (for the relevant facts on ${}_3F_2$ funtions see [2] , §4.4). We then find:

$$C_\nu^{z,k,\ell} = \frac{(k-\ell)!(\ell-z+\nu)!}{(k-z)!(\ell+z)!} \cdot$$

(4.13)

$$\cdot \sum_{\kappa_2} \frac{(-1)^{\kappa_2} (k+\ell+\kappa_2)! \quad (z-k-\nu)_{k-\ell} (-\kappa_2)_{k-\ell}}{\kappa_2!(k-\ell-\kappa_2)!(2\ell+\kappa_2+\nu+1)!(\ell+z+1)_{k-\ell}(-k-\ell-\nu-1-\kappa_2)_{k-\ell}} .$$

The only nonzero term is the one with $\kappa_2 = \ell-k$, so at last we arrive at:

$$(4.14) \qquad C_{\nu}^{z,k,\ell} = \frac{(k-\ell)!\,(k+\ell+\nu)!\,(2k)!\,(k-z+\nu)!}{((k-z)!)^2\,(k+z)!\,(2k+\nu+1)!} \;,$$

so:

$$(4.15) \qquad \pi_{\xi_z,k;k,\ell;k,\ell}(a_t) = \frac{1}{2k+1}\, e^{2t(-z+1+k+\ell)}\,{}_2F_1(-z+k+1;k+\ell+1;2k+2;1-e^{4t}).$$

It follows from (4.4) that

$$\pi_{\xi_k,z;k,\ell;k,\ell}(a_t) = \frac{1}{2k+1}\, e^{2t(z+1+k-\ell)}\,{}_2F_1(z+k+1;k-\ell+1;2k+2;1-e^{4t}),$$

so transforming (4.15) by formula (2), §2.9 of [2] we see that the respective matrix elements are indeed equal. The case $\ell \leq \pm z$ follows from the preceding case ($\ell \geq \pm z$). Indeed,

$\pi_{\xi_z,k;k,\ell;k,\ell}(a_t)$ = (by lemma 2)

$\pi_{\xi_{-z},k;k,-\ell;k,-\ell}(a_t)$ = (by the prededing case)

$\pi_{\xi_k,-z;k,-\ell;k,-\ell}(a_t)$ = (by (4.1))

$\pi_{\xi_{-k},z;k,-\ell;k,-\ell}(a_t)$ = (by lemma 2)

$\pi_{\xi_k,z;k,\ell;k,\ell}(a_t)$.

The case $|\ell| \leq |z|$ follows from the preceding cases ($|\ell| \geq |z|$).

Indeed,

$\pi_{\xi_z,k;k,\ell;k,\ell}(a_t)$ = (by lemma 2)

$$\pi_{\xi_\ell,k;k,z;k,z}(a_t) = \text{(by the preceding cases)}$$

$$\pi_{\xi_k,\ \ell;k,z;k,z}(a_t) = \text{(by (4.4))}$$

$$\frac{1}{2k+1}\, e^{2t(\ell+1+k\ z)}\, {}_2F_1(\ell+k+1;k-z+1;2k+2;1-e^{4t}) = \text{(by (4.4))}$$

$$\pi_{\xi_k,z;k,\ell;k,\ell}(a_t)$$

Summarizing we have:

<u>THEOREM 2</u>. i. *If* $\pi_{\xi_k,z}$ *is irreducible, then* $\pi_{\xi_k,z} \simeq \pi_{\xi_{-k},-z}$ *, otherwise*

$$\sigma_{\xi_k,z} \simeq \sigma_{\xi_{-k},-z} \text{ and } \tau_{\xi_k,z} \simeq \tau_{\xi_{-k},-z} .$$

ii. *If*

$$|z| < |k| \ , \ z-k \in \mathbb{Z} ,$$

then

$$\pi_{\xi_k,z} \simeq \tau_{\xi_z,k} .$$

No other equivalences exist between the various subquotient representations of the principal series.

Instead of proving explicitly the analogue of Theorem 5.10 of [6] for $SL(2,\mathbb{C})$ we shall rely on a general result due to Harish-Chandra, the famous "subquotient theorem":

Let G be a connected semi-simple Lie group with finite center, U a topologically completely irreducible representation of G on a Banach space, then U is Naimark-equivalent to a subquotient of a principal series representation of G. For this theorem see for instance [17] (theorem 5.5.1.5) where one can also find a definition of topologically complete irreducibility.

Using this a consequence of Theorem 2 is the classification of topologically completely irreducible representations of SL(2,C):

THEOREM 2'. *Every topologically completely irreducible representation of* SL(2,$\mathbb{C}$) *on a Banach space is Naimark-equivalent to exactly one of the following representations*:

i. $\pi_{\xi_k,z}$ *with* $\pi_{\xi_k,z}$ *irreducible* (*i.e.* $|z| \geq |k|$ *or* $z-k \notin \mathbb{Z}$) *and* $z \geq 0$;

ii. $\sigma_{\xi_k,z}$ ($|z| < |k|$, $z-k \in \mathbb{Z}$), $z \leq 0$.

(these are all even subrepresentations of principal series representations, cf [6], theorem 5.10. That this is true in general has been proved by Casselman)

REMARK. For K-finite or unitary representations topologically complete irreducibility is equivalent to (topological) irreducibility ([17], remark on p.305, Proposition 4.3.1.7. and Proposition 4.2.1.3).

5. UNITARIZABILITY

It is well-known (and clear from our definitions) that the conjugate contragredient (see [6],6.1) of $\pi_{\xi_k,z}$, $\pi_{\xi_k,z}^*$, is equivalent to $\pi_{\xi_k,-\bar{z}}$ so:

$$\tau^*_{\xi_k,z} \simeq \tau_{\xi_k,-\bar{z}} \, , \; \sigma^*_{\xi_k,z} \simeq \sigma_{\xi_k,-\bar{z}} \, .$$

Now a necessary condition for unitarizability of a representation τ is that it is equivalent to its conjugate contragredient τ^*. Applying the results of the preceding paragraph we find:

$$\pi_{\xi_k,z} \text{ irreducible, } \pi_{\xi_k,z}^* \simeq \pi_{\xi_k,z} \iff$$

$$(k,z) = (k,-\bar{z}) \text{ or } (k,z) = (-k,\bar{z}) \iff$$

i. z is purely imaginary, or

ii. $k = 0$, z is real, not an integer.

iii. $\tau_{\xi_k,z}{}^* \simeq \tau_{\xi_k,z} \Longleftrightarrow k = 0$.

iv. $\sigma_{\xi_k,z}{}^* \simeq \sigma_{\xi_k,z} \Longleftrightarrow k = 0$.

We treat the four cases separately. Because of the equivalences we may assume $z > 0$ if z is real, as we shall do in case *ii* and *iv*.

i. If z is purely imaginary then $\pi_{\xi_k,z}$ is unitary .

iii. $\tau_{\xi_0,z} \simeq \pi_{\xi_z,0}$, so $\tau_{\xi_0,z}$ is always unitarizible.

For the remaining cases we have to decide whether the intertwining operator between the representation and its conjugate contragredient (which is unique up to a complex scalar, according to Schur's lemma) can be chosen to be selfadjoint and positive definite. This will be a necessary and sufficient condition for unitarizability ([6], theorem 6.4). Thus we proceed by determining the various intertwining operators. If π_1 and π_2 have the same K-content: $H_{\pi_1} = \oplus H^j$, $H_{\pi_2} = \oplus H^j$ and $A:H_{\pi_1} \to H_{\pi_2}$ is an intertwining operator, then A acts on each H^j as a scalar C_j. If we fix j_0 and take $C_{j_0} = 1$, then all C_j are determined by:

(5.1) $$C_j(\pi_2(g)f_j,f_{j_0}) = (\pi_1(g)f_j,f_{j_0}) \quad (g \in G,\ f_j \in H^j,\ f_{j_0} \in H^{j_0}) .$$

Positive definiteness and selfadjointness of A is now equivalent to positivity of all C_j. To apply the above we use formula (3.11) which gives the nondiagonal matrix elements we need in (5.1).

Case ii: take $j_0 = 0$, then

$$C_j = \frac{\pi_{\xi_0,z;0,0;j,0}(a_t)}{\pi_{\xi_0,-z;0,0;j,0}(a_t)} = \frac{\overline{\pi_{\xi_0,-z;j,0;0,0}(a_{-t})}}{\pi_{\xi_0,z;j,0;0,0}(a_{-t})} =$$

$$= \frac{(1-z)_j}{(1+z)_j} \quad (j=0,1,2,\ldots) .$$

So $\pi_{\xi_0,z}$ is unitarizable if and only if $0<z<1$ (we consider only the case $z>0$)

iv. take $j_0 = 0$, then we find:

$$C_j = \frac{(1-z)_j}{(1+z)_j} \qquad (j = 0,1,\dots,z-1).$$

For $z = 1$ we find the unitarizability of the trivial onedimensional representation. For $z = 2,3,4,..$ we find that the representations $\sigma_{\xi_0,z}$ are not unitarizable.

Using theorem 2', considering the fact that two Naimark-equivalent irreducible unitary representations are unitarily equivalent ([17], Proposition 4.3.1.4), and using formula (6.8) of [6] for the redefined inner products, we see:

<u>THEOREM</u> 3. *Any irreducible unitary representation of* SL(2,ℂ) *is unitarily equivalent to one and only one of the following representations*:

1. $\pi_{\xi_k,i\lambda}$, $\lambda \geq 0$.

2. *The representation* $\pi_{\xi_0,\lambda}$ $(0<\lambda<1)$ *on* $\overline{\bigoplus_{j=0}^{\infty} H^j}$ *with respect to the inner product* < , > *defined by*

$$<f,g> = \frac{(1-z)_{j_1}}{(1+z)_{j_1}}(f,g) \qquad (f \in H^{j_1}, g \in H^{j_2})$$

(the closure being taken with respect to this inner product also).

3. *The trivial onedimensional representation.*

6. HISTORY OF THE PROBLEM

Though our approach to the representation theory of SL(2,ℂ) seems to be new the classification of representations of SL(2,ℂ) has been obtained before, by different methods. The matrix elements have also been calculated before.GELFAND and NAIMARK [5] were, in 1947 the first to classify the unitary irreducible representations of SL(2,C). NAIMARK [8] extended this result in 1954 by classifying the completely irreducible representations on a reflexive Banach space, and wrote a survey article [9]. The problem is also treated in books: NAIMARK(1958)[10], GELFAND,MINLOS and SHAPIRO

(1958)[4] , GELFAND, GRAEV and VILENKIN (1962)[3], and RUHL (1970)[11] .

Formula (2.15) for the matrix elements of the irreducible representations of SU(2) was first derived by WIGNER [18]. Formula (4.9) for these matrix elements can be found in [16].

Starting from the same integral representation as our formula (3.1), and inserting Wigner's formula STROM [14], and DUC and HIEU [1] in 1967 derived our expression (4.4) for the matrix elements of the principal series; MAKAROV and SHEPELEV [7] did the same for (4.10), using Vilenkin's formula, in 1971.

SCIARRINO and TOLLER [12] give symmetry relations for the matrix elements. STROM [13] finds a fourth order differential equation for the matrix elements.

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